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national accelerator laboratory

THE SCHOTTKY SCAN

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1. Introduction

We present an analysis of the Schottky scan on a rotating beam made of a large number of charged particles. The Schottky scan was originally intended as an application of the Schottky effect[1], and has been applied for the first time to the beams in the ISR[2]. It has also been tried in the Fermilab Booster[3] by letting the beam coast without acceleration.

We first look at the frequency manipulation of a frequency analyser (Section 2). We express the output of the analyser as function of the current in entrance. In Section 3 we discuss the current induced by the beam and the distortions due to the detection and the transport of the signal. We take into account the effects of the beam losses and of the limited duration of the observation.

The frequency spectrum of the induced current in the case of random initial conditions is investigated in Section 4. The accuracy of the scan depends only on the resolution of the frequency analyser. It is possible to measure the beam frequency distribution only if the beam losses are not correlated to the distribution.

The same analysis is applied to a coasting beam, with no-random initial conditions, in Section 5. The amplitude of the spectrum is now proportional to the total number of particles in the beam, whereas in the previous case it was to the square root of it.

Throughout the paper, we assume that the particles do not interact with each other and that no external forces are applied to the beam.

We draw the conclusion in Section 6. According to our model it is possible to perform truly Schottky scan, namely Schottky noise scan, only when the distribution of the initial condition is a random one. This is hardly satisfied in a storage ring or particle accelerator. Otherwise what is measured is essentially the structure of the beam in the phase space, and it is practically impossible to unfold the energy distribution out of a single line of the frequency spectrum.

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2. The Frequency Analyser

Let us consider the flow diagram of Fig. 1 which shows a charged beam moving down the pipe of a particle accelerator and through a charge detector. This one is connected to a frequency analyser by means of a switch S and a cable properly terminated at both ends. The switch S is fictitious and meant to replace a more complicated circuitry which triggers the frequency analyser at a desired time and over a desired period of time. The current generated by the beam at the entrance of the analyser will be denoted by j. It is a function of the time t whose origin t=0 will be set in correspondence of the closing of the switch S to let the current flow from the detector to the analyser.

The block diagram of the frequency conversion in the analyser is shown in Fig. 2. This corresponds, basically, to the Tektronix 7L12 which is actually used for "Schottky Scan" in the Fermilab Booster.

One can expand the RF input in a Fourier integral and write

$$j(t) = \int \tilde{j}(\omega)\ e^{i\omega t} d\omega$$

where here and in the following the sign of integral without limits stays for an integral from $-\infty$ to $+\infty$.

It can be easily proven with a formal construction that the current I(t) in entrance of the rectifier shown in the diagram of Fig. 2 is given by

$$I(t) = \int \tilde{j}(\omega)\ B(\Omega_1-\omega) e^{-i(\Omega_1-\Omega_2+\Omega_3-\omega)t} d\omega \tag{1}$$

where Ω_1, Ω_2 and Ω_3 are the (angular) frequencies of the local oscillators and $B(\Omega_1-\omega)$ is the resolution function of the analyser. It is zero everywhere except in a range Δ around $\Omega_1-\omega = \pm\ \omega_f$(=2.095 GHz). Δ is the resolution of the analyser.

I(t) is a 10 MHz RF current with amplitude proportional to the contained of j(t) at the frequence which is the difference between the first local oscillator frequency and 2.095 GHz. The rectifier just demodulates the 10 MHz current and one has in output a DC level proporational to the peak of I(t).

3. Analysis of the Beam Induced Current

Let us denote the total number of particles present in the beam at the time t = 0 by N. Also, let $j_s(t)$ be the current induced by the s-th particle so that

$$j(t) = \sum_{s=1}^{N} j_s(t) . \tag{2}$$

In the beam, if one considers particles as point-like charges,

$$j_s = e \sum_{n=1}^{n_s} \delta(t-t_{sn}) \tag{3}$$

where e is the charge of each particle and t_{sn} (>0) is the time when the s-th particle crosses the detector for the n-th time since the switch S has been turned on. The s-th particle crosses the detector n_s times, after that either the switch S has been reopened and the measurement interrupted or the particle is somehow lost, whichever occurs first.

The Fourier expansion of (3) gives easily

$$j_s = \frac{e}{2\pi} \sum_{n=1}^{n_s} \int e^{i\omega(t-t_{sn})} d\omega. \tag{4}$$

So far, as also shown in Fig. 1, a particle has been represented by a delta function in the beam, (A). But the distribution of the current induced on a perfectly conductive wall, (B), has the shape of a bell[4] with a full width at half of the maximum equal to the pipe size divided by the ratio of the particle total energy to the rest energy.

In case the detector is made of a perfectly conductive cylindrical plate surrounding the beam, the current in exit, (C), is the integration, at a given time t, of the induced current over the detector length. In addition, the detector itself introduces a distortion which can be characterized by a cut-off frequency.

Further distortions can be introduced by the transport of the signal from the detector to the frequency analyser, (D), which mainly consists of a long matched cable.

Thus it is convenient to modify eq.(4) as follows

$$j_s = \frac{e}{2\pi} \sum_{n=1}^{n_s} \int g_{sn}(\omega)\, e^{i\omega(t-t_{sn})}\, d\omega \tag{5}$$

where g_{sn} may depend on the energy of the s-th particle and includes all the effects mentioned above.

One can derive the beam induced current at the entrance of the frequency analyser by combining (2) and (5). One has

$$j(t) = \frac{e}{2\pi} \sum_{s=1}^{N} \sum_{n=1}^{n_s} \int g_{sn}(\omega)\, e^{i\omega(t-t_{sn})}\, d\omega\,. \tag{6}$$

We shall consider only the following case: all the particles have constant velocities, and

$$t_{sn} = t_s + (n-1)T_s$$

where t_s is the time of the first traversal of the detector ($t_s>0$) and T_s is a constant which depends only on the velocity of the particle. In this case (6) becomes

$$j(t) = \frac{e}{2\pi} \sum_{s=1}^{N} \int g(\omega,T_s)\, A(\omega,n_s,T_s)\, e^{i\omega(t-t_s)}\, d\omega \tag{7}$$

where $g(\omega,T_s)$ replaces $g_{sn}(\omega)$ and

$$A(\omega,n_s,T_s) = \sum_{n=1}^{n_s} e^{-i\omega(n-1)T_s}$$

$$= e^{-i\frac{n_s-1}{2}T_s\omega}\ \frac{\sin\frac{n_s T_s \omega}{2}}{\sin\frac{T_s\omega}{2}}\,. \tag{8}$$

4. The Frequency Analysis of the Beam Induced Current

The Fourier transform $\tilde{j}(\omega)$ of the beam induced current $j(t)$ at the entrance of the frequency analyser is obviously given by the function which multiplies $e^{i\omega t}d\omega$ under the sign of integral at the righthand side of (7), namely

$$\tilde{j}(\omega) = \frac{e}{2\pi} \sum_{s=1}^{N} g(\omega, T_s)\ A(\omega, n_s, T_s)\ e^{-i\omega t_s}$$

which has to be inserted at the righthand side of (1). The integration over ω can be done with the method of the residuals. For this purpose we assume that $B(\Omega_1-\omega)$ and $g(\omega,T_s)$ have no singularities in ω. Then the only singularities are those of the function $A(\omega,n_s,T_s)$ given by (8).

One obtains

$$I(t) = 2 \sum_{k=1}^{\infty} \sum_{s=1}^{N} \frac{e}{T_s} \left| g\left(\frac{2\pi k}{T_s}, T_s\right) B\left(\Omega_1 - \frac{2\pi k}{T_s}\right) \right| H(t-t_s, n_s, T_s) \times$$

$$\times \cos\left(\omega_o t + 2\pi k \frac{t_s}{T_s} + \phi_{ks}\right) \qquad (9)$$

where

$$H(t,n,T) = 1, \qquad \text{for } -T/2 < t < nT - T/2$$
$$= 0, \qquad \text{otherwise}$$

$$\omega_o = \Omega_1 - \Omega_2 + \Omega_3 - 2\pi \frac{k}{T_s} = 2\pi \times 10 \text{ MHz}$$

and ϕ_{ks} is some phase angle which does not depend on t_s.

Let us take the square of both sides of (9) that we shall denote by I^2. It is made of four summations

$$\sum_{k_1=1}^{\infty} \quad \sum_{k_2=1}^{\infty} \quad \sum_{s_1=1}^{N} \quad \sum_{s_2=1}^{N} \quad .$$

To perform the summations on s_1 and s_2, let us introduce a 4-dimensional space of coordinates T_1, n_1, T_2 and n_2 and let us divide it in identical cells, small enough so that the four variables change so little inside that they can be kept constant, but large enough to accommodate a large number of particles. Thus the double summation can be performed by summing first over all the particles in each individual cell, of coordinates (T_{1c}, n_{1c}, T_{2c}, n_{2c}), and then over all the cells. If we assume that t_s's are not correlated to the n_s's and T_s's, it is obvious that one has to deal with partial summations of the kind

$$S = \underbrace{\sum_{s_1} \sum_{s_2}}_{\text{over one cell}} \cos\left(\omega_o t + 2\pi k_1 \frac{t_{s1}}{T_{1c}} + \phi_{k_1 c}\right) \cos\left(\omega_o t + 2\pi k_2 \frac{t_{s2}}{T_{2c}} + \phi_{k_2 c}\right) .$$

We shall assume that the distribution of the t_s's is completely random over the range (0, T_c). We want to estimate the expectation value of S as the average over an ensemble of $M \to \infty$ beams owing the same properties but with different random initial conditions.

It is easily proved that the terms with $s_1 \neq s_2$ are identically zero in the limit $M \to \infty$. The only contribution comes from the terms $s_1 = s_2$ which correspond to the same particle, and they are located only on the diagonal cells where $n_{1c} = n_{2c} = n_c$ and $T_{1c} = T_{2c} = T_c$. Moreover this contribution is different from zero and equal to $N_c/2$, with N_c the actual number of particles in the diagonal cell, only if $k_2 = k_1 = k$. Then one has from (9)

$$I^2 = 2 \sum_{k=1}^{\infty} \sum_{c} \frac{e^2}{T_c{}^2} N_c \, H(t,n_c,T_c) \times \left| g\left(\frac{2\pi k}{T_c},T_c\right) B\left(\Omega_1 - \frac{2\pi k}{T_c}\right)\right|^2 . \tag{10}$$

Thus the power spectrum is made of infinitely many lines all centered around frequencies which are integer times the main revolution frequency. The width and height of each line depends on the resolution of the frequency analyser, on the distortion the current suffers from the beam to the frequency analyser, on the beam loss and on the spread in the beam.

In a typical scan, one would vary Ω_1 to pick up only one mode k small enough so that

$$g\left(\frac{2\pi k}{T},T\right) \sim 1 . \tag{11}$$

On the other side it is desirable to choose a high value of k so that the resulting width of $I(\Omega_1)$ is large enough compared to the resolution Δ of the analyser, but not too large to avoid overlapping of neighboring distributions at different k.

By introducing a continuous distribution function $\rho(T,n)$ such that

$$\iint \rho(T,n) \quad dT\, dn = 1$$

one obtains from (10)

$$I(\Omega_1) = \left\{ 2N \iint \frac{e^2}{T^2} H(t,n,T) \left| B(\Omega_1 - \frac{2\pi k}{T}) \right|^2 \rho(T,n)\, dn\, dT \right\}^{1/2} .$$

In the special case of monoenergetic beam with no losses, one has

$$I(\Omega_1) = \sqrt{2N}\, \frac{e}{T} \left| B(\Omega_1 - \frac{2\pi k}{T}) \right| . \tag{12}$$

If one plots I versus Ω_1 one finds a line centered at $\Omega_1 = \omega_f + 2\pi k/T$ having a width Δ. This replaces the ideal line with zero width which corresponds to the beam. In any case the accuracy of the measurement depends only on the resolution of the analyser. The overall frequency resolution is not affected by the length of the observation time t spent for a specific value of Ω_1.

In case of no beam losses one has

$$I(\Omega_1) = \left\{ 2N \int \frac{e^2}{T^2} \left| B(\Omega_1 - \frac{2\pi k}{T}) \right|^2 \rho(T)\, dT \right\}^{1/2} \tag{13}$$

and the plot of I^2 versus Ω_1 gives the distribution $\rho(T)$. For instance, if k is large enough, one can disregard the accuracy of the analyser and has

$$N \rho(T) = \frac{\pi k}{e^2} I^2(\Omega_1) \quad . \tag{14}$$

One would obtain the same result also in the case of beam loss which is not correlated to the frequency distribution. In this case one can write $\rho(T,n) = \rho(T)\ \sigma(n)$ and make use of (14) after having replaced N with the number of particles N(t) survived at the time t. But in the case the loss is correlated to the frequency distribution, then it is not possible to unfold this distribution out.

5. Frequency Analysis of a Coherent Beam

In this section we shall consider the case that the distribution of the initial conditions of the particles in the beam is not random. We introduce the longitudinal phase space of the canonical variables Θ and W, Θ being actually the angle around the reference orbit. Let $F(\Theta,W)$ be the particle distribution at the time t_o prior to the instant t=0 when the switch S of Fig. 1 is turned on. We take

$$\iint F(\Theta,dW)\ d\Theta dW = 1 \ .$$

In the case there are no interactions among the particles, and no external forces of any sort are applied to the beam, the distribution at a later time t is given by

$$F(\Theta - \Omega(t-t_o),\ W)$$

where Ω is the angular revolution frequency of a particle with variable W. The current associated to the beam at the location of the charge detector (Θ=0), then, is

$$j(t) = Ne\int \Omega\ F(-\Omega(t-t_o),W)\ dW \ .$$

Since $F(\Theta,W)$ is a periodic function of Θ with period 2π we can also write

$$j(t) = Ne \sum_k \int \Omega F_k(W) \quad e^{-ik\Omega(t-t_o)} dW$$

with

$$F_k(W) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(\Theta,W) \; e^{ik\Theta} d\Theta \quad .$$

The Fourier expansion of this current which we assume to last from the time $t=0$ to the time t_1, gives

$$j(t) = \int \tilde{j}(\omega) \; e^{i\omega t} d\omega$$

where

$$\tilde{j}(\omega) = i\frac{Ne}{2\pi} \sum_k \int \Omega \; F_k(W) \; e^{ik\Omega t_o} \; \frac{e^{-i(k\Omega+\omega)t_1}-1}{k\Omega+\omega} \, dW \quad .$$

Introducing again the transmission function $g(\omega,W)$ as it was done in Section 3, we have for the current at the entrance of the frequency analyser

$$j(t) = \int \tilde{j}(\omega) \; g(\omega,W) \; e^{i\omega t} d\omega \quad .$$

The integration over ω at the righthand side of (1) can again be done with the method of the residuals. We assume again that $B(\Omega_1-\omega)$ and $g(\omega,W)$ have no singularities.

We obtain for $0<t<t_1$

$$I(\Omega_1) = Ne \sum_{k=1}^{\infty} \int \Omega \left| F_k(W)\ g(k\Omega,W) \times B(\Omega_1 - k\Omega) \right| dW . \tag{15}$$

One can again retain only one particular value of k such that (11) applies, and

$$I(\Omega_1) = Ne \int \Omega \left| F_k(W)\ B(\Omega_1 - k\Omega) \right| dW. \tag{16}$$

In particular for a monoenergetic beam we can take $F_k(W) = \alpha_k\ \delta(W)$ and have

$$I(\Omega_1) = Ne\ \ \Omega\alpha_k \left| B(\Omega_1 - k\Omega) \right| \tag{17}$$

which corresponds to a line centered at $\Omega_1 = k\Omega + \omega_f$ having a width Δ and an amplitude proportional to α_k.

In the more general case it is not possible to unfold the energy distribution out of (16), unless the initial distribution function $F(\Theta,W)$ is separable.

Finally, let us observe that the spectrum (15) does not depend on the time t_o during which the beam is let to "debunch" before to start the observation.

6. Conclusions

We have calculated the frequency spectrum of a rotating beam made of a large number of particles in two different cases: (i) At an initial time prior to the starting of the observation, the distribution of the particles in the longitudinal phase space (Θ,W)

is completely random. (ii) At the same initial time the distribution function of the particles in the longitudinal phase space is not-random and is given by an analytical function $F(\Theta,W)$.

In both cases we assumed that, from that initial time on and during all the observation, the particles were not interacting each other and no external forces were applied to them. Therefore, in the first case, the distribution remains random also during the observation period, whereas in the second case, the bunching structure of the beam will be preserved in the (Θ,W) phase space.

The frequency analysis of the beam made with the technique known as Schottky scan, is sensitive to the distribution in the longitudinal phase space. Furthermore, as we have seen in the previous section, the result of the analysis does not depend on the time one waits to let the beam "debunch".

Our main result is given by eq.s(13) and (16) which are the beam spectra respectively for the case (i) of random initial conditions and for the case (ii) of non-random initial conditions. The special formula for the case of a monoenergetic beam are given by eq.s (12) and (17).

Because the beam Schottky "noise" is proportional to the square root of the number of the particles, it usually provides a weaker signal, whereas the coherence of the beam, being proportional to the number of the particles, usually provides a stronger signal.

The two signals would equal when the coherence factor $2\pi\alpha_k$ satisfies the relation

$$2\pi\alpha_k = \sqrt{2/N} .$$

For instance in the case of a beam made of 10^{14} particles (like in the ISR), truly Schottky noise scan is possible only when $2\pi\alpha_k << 10^{-7}$. A condition which we believe is rather hard to satisfy.

References

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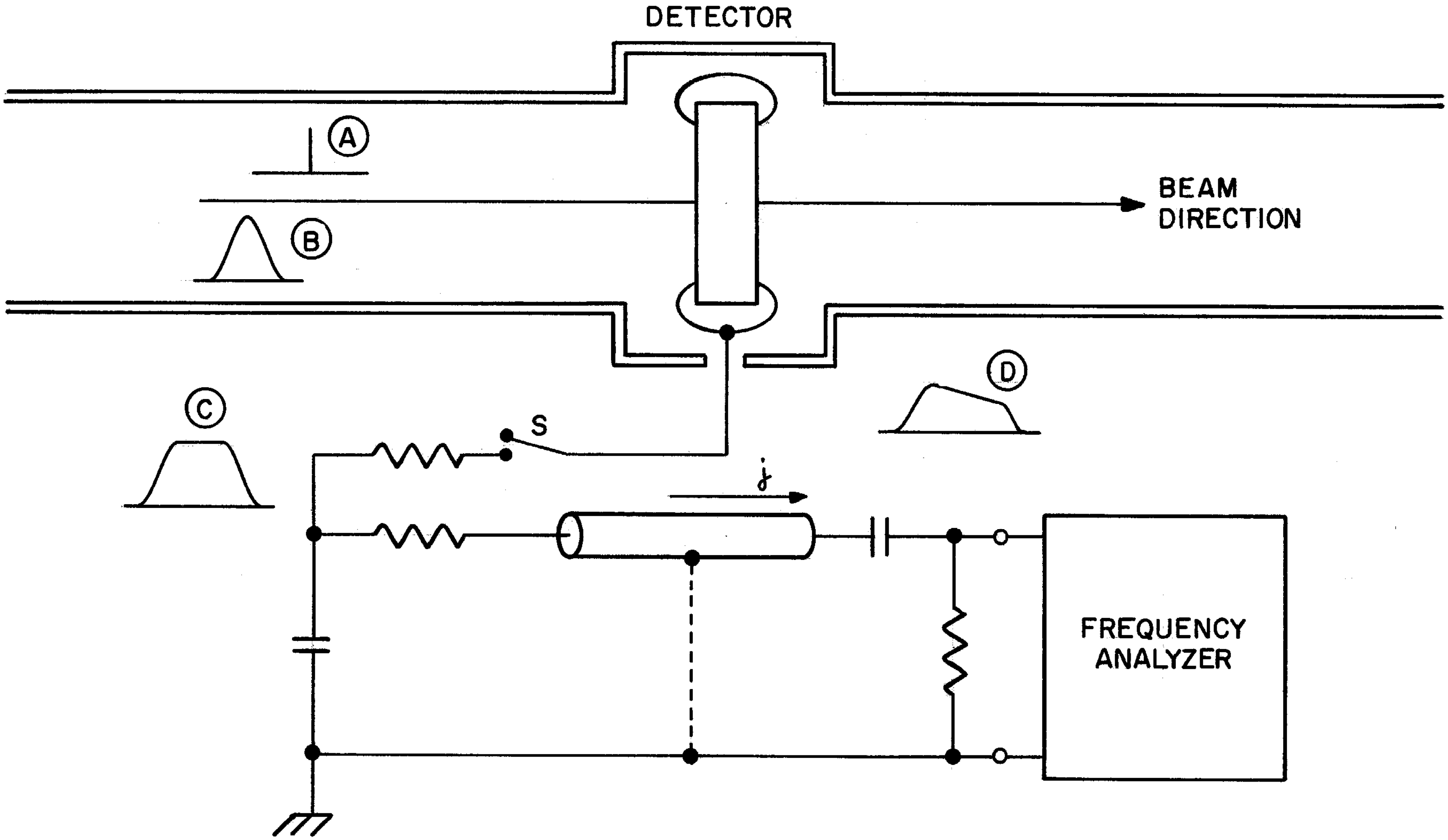

FIG. I

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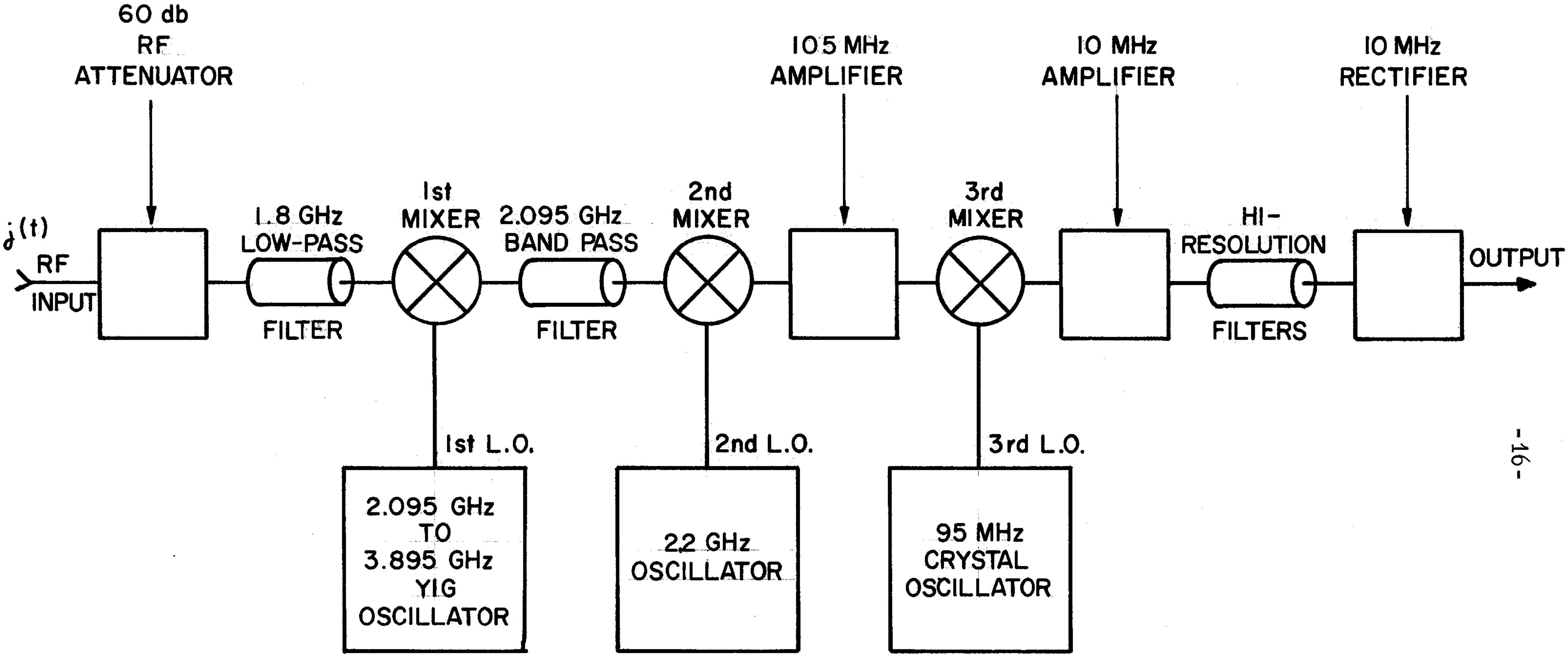
60 db
RF
ATTENUATOR
$j(t)$
RF
INPUT
1.8 GHz
LOW-PASS
FILTER
1st
MIXER
2.095 GHz
BAND PASS
FILTER
2nd
MIXER
10.5 MHz
AMPLIFIER
3rd
MIXER
10 MHz
AMPLIFIER
HI-
RESOLUTION
FILTERS
10 MHz
RECTIFIER
OUTPUT
1st L.O.
2.095 GHz
TO
3.895 GHz
YIG
OSCILLATOR
2nd L.O.
2.2 GHz
OSCILLATOR
3rd L.O.
95 MHz
CRYSTAL
OSCILLATOR

FIG. 2